Backpaper - Partial Differential Equations (2022-23)
Time: 3 hours.
Attempt all questions, giving proper explanations.

1. Give an example to show that non-uniqueness might hold for differential equations of the form $u'(t) = f\big(u(t)\big),\ u(t_0) = u_0$, with continuous $f : \mathbf{R} \to \mathbf{R}$, unless some additional assumptions are made on f. **[4 marks]**

2. Find the general solution $u : \mathbf{R} \to \mathbf{R}$ of $u'' + 2u' + u = e^{-x}$. **[6 marks]**

3. Find the solution of the one dimensional heat equation $\partial_t u = \partial_{xx} u$ on the spatial interval $[0, \pi]$, with boundary conditions $u(t, 0) = u(t, \pi) = 0$, and initial profile $u(0, x) = f(x)$ satisfying the boundary conditions. Clearly state any assumptions you might make in solving the problem. **[7 marks]**

4. Solve the equation $(1 + x^2)u_x + u_y = 0$ for $u : \mathbf{R}^2 \to \mathbf{R}$. Sketch some of the characterisitc curves. **[6 marks]**

5. Suppose $u^{(i)}$, $i = 1, 2$ satisfy the one dimensional heat equation $u_t^{(i)} - u_{xx}^{(i)} = f(x, t)$ with boundary conditions $u^{(i)}(0, t) = g(t),\ u^{(i)}(1, t) = h(t)$ for $t > 0$, and initial condition $u^{(i)}(x, 0) = \phi^{(i)}(x)$ for $0 < x < 1$. Show that

$$\max_{0 \le x \le 1} \left| u^{(1)}(x,t) - u^{(2)}(x,t) \right| \le \max_{0 \le x \le 1} \left| \phi^{(1)}(x) - \phi^{(2)}(x) \right| \quad \text{for } t > 0.$$

   [5 marks]

6.
   - Write down the fundamental solution of the heat equation: $u_t - \Delta u = 0$ on $\mathbf{R}^n \times (0, \infty)$. **[3 marks]**
   - Write down the fundamental solution of Laplace's equation: $\Delta u = 0$ on $\mathbf{R}^n$ **[3 marks]**

7. Let $u : \mathbf{R}^n \to \mathbf{R}$ be a twice continuously differentiable function which satisfies

$$u(\mathbf{x}) = ⨍_{\partial B(\mathbf{x},r)} u(\mathbf{y})dS(\mathbf{y})$$

   for all $\mathbf{x}$ and $r > 0$, where the right hand side is the average integral of u over the boundary of the ball $B(\mathbf{x}, r)$.

   - Show that
   $$u(\mathbf{x}) = ⨍_{B(\mathbf{x},r)} u(\mathbf{y})d\mathbf{y}, \qquad \textbf{[5 marks]}$$
   - What PDE does u satisfy? **[2 marks]**

8. Let $H : \mathbf{R}^n \to \mathbf{R}$ be convex.
   - Write down the definition of the Legendre transform of H (denoted as H^*). **[2 marks]**
   - We say $\mathbf{q}$ belongs to the subdifferential of H at $\mathbf{p}$, written $\mathbf{q} \in \partial H(\mathbf{p})$, if

   $$H(\mathbf{r}) \ge H(\mathbf{p}) + \mathbf{q} \cdot (\mathbf{r} - \mathbf{p}), \qquad \text{for all } \mathbf{r} \in \mathbf{R}^n.$$

   Prove $\mathbf{q} \in \partial H(\mathbf{p})$ if and only if $\mathbf{p} \in \partial L(\mathbf{q})$ if and only if $\mathbf{p} \cdot \mathbf{q} = H(\mathbf{p}) + L(\mathbf{q})$, where $L = H^*$. **[7 marks]**